1(i) $\vec{r}(t) = (x(t), y(t), z(t))$

$$= \left(\tfrac{t^2}{2}, \ln t, \sqrt{2}\,t\right)$$

$\vec{r}'(t) = \left(t, \tfrac{1}{t}, \sqrt{2}\right)$

Length of the curve $= \int_1^2 |\vec{r}'(t)|\,dt$

$$= \int_1^2 \sqrt{t^2 + \left(\tfrac{1}{t}\right)^2 + 2}\;dt$$

$$= \int_1^2 \frac{1}{t}\sqrt{t^4 + 2t^2 + 1}\;dt$$

$$= \int_1^2 \frac{1}{t}(t^2+1)\;dt$$

$$= \left[\tfrac{t^2}{2} + \ln t\right]_{t=1}^{2}$$

$$= 2 + \ln 2 - \tfrac{1}{2} = \tfrac{3}{2} + \ln 2.$$

1(ii) Let $\vec{r}(t) = (x(t), y(t), z(t))$

$$= (3t\sin t, 3t\cos t, 2t^2)$$

$\vec{r}'(t) = (3\sin t + 3t\cos t, 3\cos t - 3t\sin t, 4t)$

Length of the curve $= \int_0^{4/5} |\vec{r}'(t)|\,dt$

$$= \int_0^{4/5} \sqrt{(3\sin t + 3t\cos t)^2 + (3\cos t - 3t\sin t)^2 + (4t)^2}\;dt$$

$$= \int_0^{\frac{4}{5}} \sqrt{9 + 9t^2 + 16t^2}\;dt = \int_0^{4/5} \sqrt{25t^2+9}\;dt$$

$$= \int_0^{\frac{4}{3}} 3\sqrt{s^2+1} \left(\frac{3}{5}\, ds \right)$$

let $t = \frac{3}{5} s$

$$= \frac{9}{5} \int_0^{\frac{4}{3}} \sqrt{s^2+1}\, ds$$

$$= \frac{9}{5} \int_0^{\tan^{-1}(\frac{4}{3})} \sqrt{\tan^2\theta + 1}\, \sec^2\theta\, d\theta$$

let $s = \tan\theta$

$ds = \sec^2\theta\, d\theta$

$$= \frac{9}{5} \int_0^{\tan^{-1}(\frac{4}{3})} \sec^3\theta\, d\theta$$

$$\int \sec^3\theta\, d\theta = \int (\tan^2\theta + 1) \sec\theta\, d\theta$$

$$= \int \sec\theta \tan^2\theta\, d\theta + \int \sec\theta\, d\theta$$

known result

$$= \int \tan\theta\, d(\sec\theta) + \boxed{\ln|\sec\theta + \tan\theta|}$$

by parts

$$= \sec\theta \tan\theta - \int \sec\theta\, d(\tan\theta) + \ln|\sec\theta + \tan\theta|$$

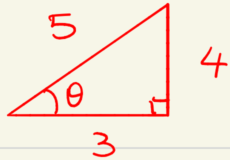
$$= \sec\theta \tan\theta - \int \sec^3\theta\, d\theta + \ln|\sec\theta + \tan\theta|$$

$\therefore$ $$\int \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \right) + C$$

$\therefore$ Length of the curve $$= \frac{9}{5} \int_0^{\tan^{-1}(\frac{4}{3})} \sec^3\theta\, d\theta$$

$$= \frac{9}{10} \Big[\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \Big]_{\theta=0}^{\tan^{-1}(\frac{4}{3})}$$

For $\theta = \tan^{-1}\left(\frac{4}{3}\right)$,



,

$\sec\theta = \frac{5}{3}$

$\tan\theta = \frac{4}{3}$

$$= \frac{9}{10}\left[\frac{20}{9} + \ln 3 \right] = 2 + \frac{9}{10}\ln 3$$

2.

$$\frac{d}{dt}\left(\vec{r}(t) \times \vec{v}(t)\right) = \vec{r}'(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{v}'(t)$$

$$= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t)$$

$$= \vec{0} + \vec{r}(t) \times \frac{1}{m}\vec{F}$$

$$= \vec{r}(t) \times \underbrace{\frac{-GM}{|\vec{r}(t)|^3}}_{\text{scalar}}\vec{r}(t)$$

$$= \vec{0} \quad \left(\because \vec{r}(t) \times \vec{r}(t) = 0\right)$$

$\therefore$ $\vec{r}(t) \times \vec{v}(t)$ is a constant vector.

3.

$$\vec{r}(t) = \langle r(t)\cos\theta(t), r(t)\sin\theta(t), 0 \rangle$$

$$\vec{v}(t) = \vec{r}'(t)$$

$$= \langle r'(t)\cos\theta(t) - r(t)\sin\theta(t)\,\theta'(t),$$
$$r'(t)\sin\theta(t) + r(t)\cos\theta(t)\,\theta'(t), 0 \rangle$$

$$= r'(t)\langle \cos\theta(t), \sin\theta(t), 0 \rangle$$
$$- r(t)\theta'(t)\langle \sin\theta(t), -\cos\theta(t), 0 \rangle$$

Same direction

$\vec{r}(t) \times \vec{v}(t)$

$= r(t) \underline{\langle \cos\theta(t), \sin\theta(t), 0 \rangle} \times \left(r'(t) \underline{\langle \cos\theta(t), \sin\theta(t), 0 \rangle} \right.$

$\left. - r(t)\theta'(t) \langle \sin\theta(t), -\cos\theta(t), 0 \rangle \right)$

$= -r^2(t)\,\theta'(t) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta(t) & \sin\theta(t) & 0 \\ \sin\theta(t) & -\cos\theta(t) & 0 \end{vmatrix}$

$= r^2(t)\,\theta'(t)\,\hat{k}$

4. Using the notation in problem 3.

for an orbit, position of the planet is completely determined by its angle. θ

Let $\vec{s}(\theta)$ be the position of the planet when the angle is θ. Then, $\vec{s}(\theta(t)) = \vec{r}(t)$.

Consider a time interval $[t_1, t_2]$,

WLOG, we may assume that $0 < \theta(t_1)$ and

$\theta(t_2) - \theta(t_1) < 2\pi$

The area swept out by the planet within $[t_1, t_2]$

$$= \frac{1}{2}\int_{\theta(t_1)}^{\theta(t_2)} |\vec{s}(\theta)|^2 \, d\theta$$

$$= \frac{1}{2}\int_{t_1}^{t_2} |\vec{s}(\theta(t))|^2 \, \theta'(t) \, dt$$

[by change of variable, $\theta = \theta(t)$]

$$= \frac{1}{2}\int_{t_1}^{t_2} |\vec{r}(t)|^2 \, \theta'(t) \, dt$$

(by Q3)

$$= \frac{1}{2}\int_{t_1}^{t_2} r(t)^2 \, \theta'(t) \, dt = \int_{t_1}^{t_2} |\vec{r}(t) \times \vec{v}(t)| \, dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2} C \, dt$$

(by Q2, $\vec{r}(t) \times \vec{v}(t)$ is a constant vector)

$$= \frac{1}{2} C (t_2 - t_1)$$

$\therefore$ The area only depends on the length of time interval, $t_2 - t_1$.

5.

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

$$x(t) = r(t)\cos\theta(t) \;;\; x'(t) = r'(t)\cos\theta(t) - r(t)\sin\theta(t)\,\theta'(t)$$
$$y(t) = r(t)\sin\theta(t) \;;\; y'(t) = r'(t)\sin\theta(t) + r(t)\cos\theta(t)\,\theta'(t)$$

$$x'(t)^2 + y'(t)^2 = r'(t)^2 + r(t)^2\,\theta'(t)^2$$

$$\therefore \quad L = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$
$$= \sqrt{r'(t)^2 + (r(t)\theta'(t))^2 + z'(t)^2}$$

6. Let $\vec{\gamma}(t) = (t, t^2, t^3)$. $\gamma'(t) = (1, 2t, 3t^2)$

$\therefore$ The tangent vector of the curve at $(1,1,1)$

is $\gamma'(1) = (1,2,3)$ $[\because \gamma(1) = (1,1,1)]$

Equation of the normal plane is

$$x + 2y + 3z = 6$$

7. $$\vec{v}'(t) = \vec{a}(t) = \langle e^{-3t}, t, \sin t \rangle$$

$$\vec{v}(t) - \vec{v}(0) = \left\langle \int_0^t e^{-3s}\,ds, \int_0^t s\,ds, \int_0^t \sin s\,ds \right\rangle$$
$$= \left\langle \frac{1}{3}(1 - e^{-3t}), \frac{t^2}{2}, 1 - \cos t \right\rangle$$

$$\vec{v}(t) = \left\langle \frac{13}{3} - \frac{1}{3}e^{-3t}, \frac{t^2}{2} - 2, 5 - \cos t \right\rangle$$

$$\vec{r}'(t) = \vec{v}(t)$$

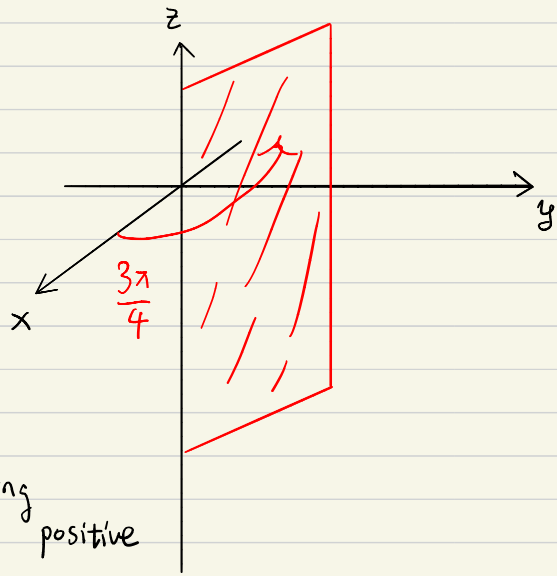
$$\vec r(t) - \vec r(0) = \left\langle \int_0^t \frac{13}{3} - \frac{1}{3}e^{-3s}\,ds,\ \int_0^t \frac{s^2}{2} - 2\,ds,\ \int_0^t 5 - \cos s\,ds \right\rangle$$

$$= \left\langle \frac{13}{3}t + \frac{1}{9}(e^{-3t} - 1),\ \frac{t^3}{6} - 2t,\ 5t - \sin t \right\rangle$$

$$\therefore \vec r(t) = \left\langle \frac{13}{3}t + \frac{1}{9}e^{-3t} - \frac{1}{9},\ \frac{t^3}{6} - 2t + 4,\ 5t - \sin t - 2 \right\rangle$$

8. (i)

$$\theta = \frac{3\pi}{4}$$



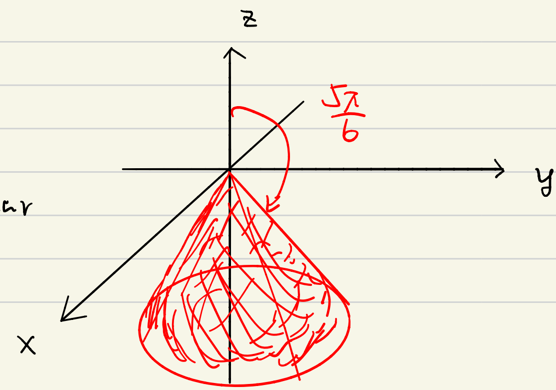
(surface)

It is a plane making angle $3\pi/4$ with the positive x-axis.

(ii)

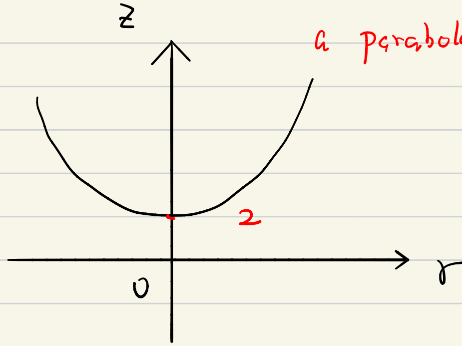
$$\phi = \frac{5\pi}{6}$$

It is a right circular cone (surface)

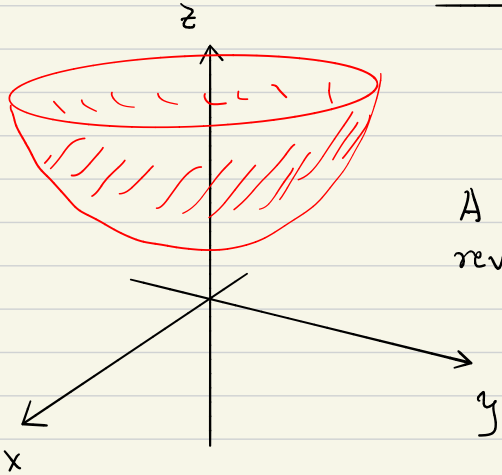


(iii) $z - 4r^2 = 2$

$z = 2 + 4r^2$

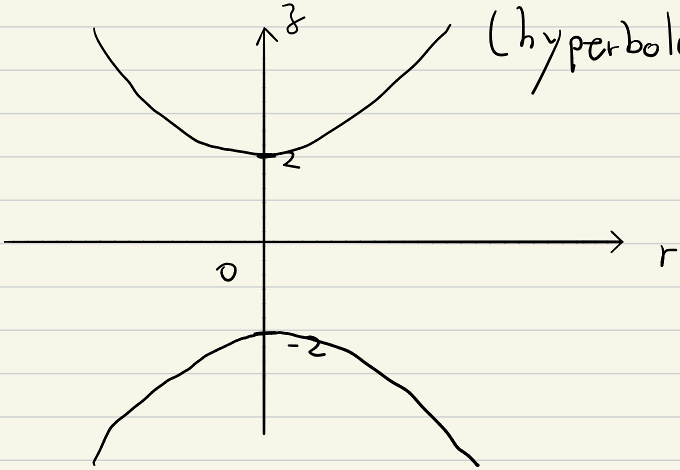


a parabola

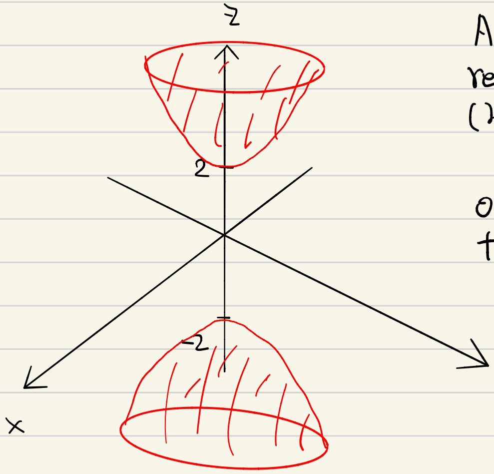


A surface obtained by revolution of the curve $z = 2+4y^2$ on the zy-plane around z-axis

(iv) $z^2 - 2r^2 = 4$



(hyperbola)



A surface obtained by revolution of the curve (hyperbola)

$$z^2 - 2y^2 = 4$$

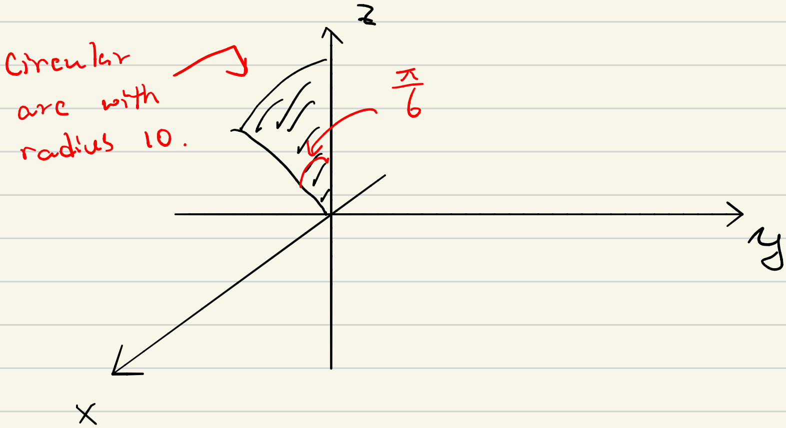
on the zy-plane around the z-axis.

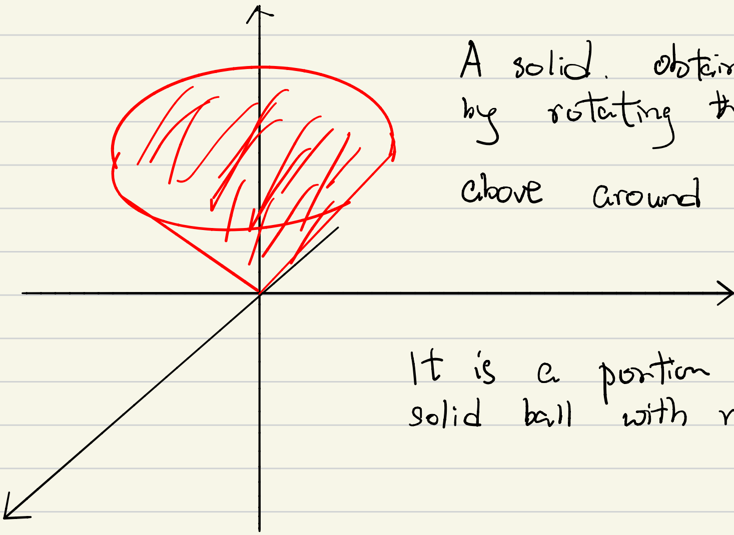
(v)

$$0 \le \phi \le \frac{\pi}{6}, \quad 0 \le \rho \le 10$$

For $0 \le \phi \le \frac{\pi}{6}$, $0 \le \rho \le 10$, $\theta = 0$,

it is a surface on the xz-plane





A solid obtained by rotating the surface above around the z-axis.

It is a portion of the solid ball with radius 10.

9. Along the path $x=0$,

$$\lim_{y \to 0} \frac{0 \cdot y^2}{0^2 + y^4} = 0$$

Along the path $x = y^2$, and approaching $(0,0)$,

$$\lim_{y \to 0} \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{1}{2}$$

Since different paths give different limits,

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+y^4} \quad \text{DNE.}$$

10. Let $\varepsilon > 0$,

$$|f(x,y) - 8|$$

$$= |3x + 2y - 8|$$

$$= |3(x-2) + 6 + 2(y-1) + 2 - 8|$$

$$= |3(x-2) + 2(y-1)|$$

$$\leq \sqrt{3^2 + 2^2} \cdot \sqrt{(x-2)^2 + (y-1)^2}$$

(Cauchy - Schwarz inequality)

Assignment 1 Q3(i)

$$= \sqrt{13}\sqrt{(x-2)^2 + (y-1)^2}$$

$\therefore$ if we put $\delta = \frac{\varepsilon}{\sqrt{13}}$, and if

$$0 < |(x,y) - (2,1)| = \sqrt{(x-2)^2 + (y-1)^2} < \delta,$$

then

$$|f(x,y) - 8| \leq \sqrt{13} \cdot \sqrt{(x-2)^2 + (y-1)^2}$$

$$< \varepsilon$$

#